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Math 5010

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Cavalieri's Principle

Cavalieri's principle was named after Bonaventura Cavalieri born in Milan Italy. his interest in mathematics was heavily influenced by Euclid’s Elements and a meeting that he had with Galileo Cavalieri Applied to be the chair of mathematics at several Universities but was rejected many times due to him being too young and because his religious beliefs were looked down on. When Cavalieri finally did become a professor of mathematics he published the method of indivisibles which was a precursor to the Cavalieri principle but was put under scrutiny and rejected by scholars. Later Cavalieri would write "Six Geometrical Exercises" where he better explained his ideas, after which his ideas finally started to catch on.

Cavalieri’s Principle is easily defined in both 2 and three dimensions. What is Cavalieri's Principle? Put simply if 2 shapes have the same height and if you are able to slice both of them at any corresponding height and every such cross section is equal the two shapes have the same area if in 2 dimensions and the two shapes have the same area if dealing with three dimensions. This allows you to compare strange shapes to standardized shapes with known equations.

Probably the most basic example of this principle relates a cylinder to a rectangular prism. A rectangular prism with the given dimensions of 3,3,5 and a cylinder with a height of 5 and a radius of $\frac{3}{\sqrt{\pi}}$ we can use the equations to show that the 2 shapes have the same volume $3 \bullet 3 \bullet 5 = 45$ or $5\pi(\frac{3}{\sqrt{\pi}})^2 = 5\pi\frac{9}{\pi} = 45$. Another valid way for us to show that the two shapes have the same volume is to use Cavalieri's Principle. If we put the two shapes side by side so that the height is 5 for each shape. In regards to the cylinder for every possible cross section parallel to the base of the cylinder we get a circle with radius $\frac{3}{\sqrt{\pi}}$ The area of any of these particular cross sectioned circles is 9 units squared. For the rectangular prism each cross section defined the same way as we defined it for the cylinder will give a 3 by 3 square which has an area of 9 units squared. We can rightly make the claim: because the two shapes have the same height and for every cross section normal to the height the two shapes have the same area the two shapes must also have the same volume by Cavalieri's Principle.

Using Cavalieri's Principle to compare cylinders and rectangular prisms isn't very interesting and feels very trivial (because it is) ;however, we can use this principle in a much more interesting way. One simple example of this is: given shape defined by the formula $x = 2\sqrt{1 - y^2}$ and the y-axis finds the area . we can use calculus to solve the area of the first shape by integrating from -1 to 1 in terms of y which is a valid yet time consuming way to find that the area is π. Another way to find the area is to simply use Cavalieri's Principle. We can compare our shape to a circle defined as $x^2 + y^2 = 1$.

Given any height has a relation to y we can show that .

$x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2 \Rightarrow x = \pm\sqrt{1 - y^2}$ because the 2 x values on the circle at any height y is $\pm\sqrt{1 - h^2}$ that implies that length of the cross section equals $\sqrt{1 - y^2} - (-\sqrt{1 - y^2}) = \sqrt{1 - y^2} + \sqrt{1 - y^2} = 2\sqrt{1 - y^2}$ because the length of the cross sections of the circle is equivalent to the cross sections of the shape above we can determine that the shape we defined by $x = 2\sqrt{1 - y^2}$ and the y-axis has the same area as the unit circle (a circle of radius 1).

The area for any given cone is $\frac{1}{3} base \times height$. where base is simply the area of the base. We know confidently that every cone follows this equation because of Cavalieri's Principle. to show that any cone with the same height and the same base area will have the same volume is actually a simple matter. Given any cone with a base of area b and a height of h we can find the scaling factor at any height. The scaling factor equals (h-k)/h where k is the height of the cross section. The formula for the area of any scaled shape equals (the area of the original shape) *(the scale Factor)^2 therefore the area of any given cross section normal to the height and parallel to the base at height k is: $b\frac{(h-k)^2}{h^2}$. because the shape of the base has no effect on the formula for scaling we can use Cavalieri's Principle to conclude that the formula for any cone is $\frac{bh}{3}$.

Citations

Khan Academy. (n.d.). *Cavalieri's principle in 3D (article)*. Khan Academy. https://www.khanacademy.org/math/geometry/hs-geo-solids/xff63fac4:hs-geo-cavalieri-s-principle/a/cavalieri-s-principle-in-3d

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